CHRISTMAS COMBINATORICS

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Problem 1

Alicia found six little pinecones to decorate three mini Christmas wreaths. She could have placed two pinecones on each wreath, but she liked to do things a little bit differently so she placed one pinecone on the first wreath, two pinecones on the second wreath, and three pinecones on the third wreath.
In how many different ways can one divide six pinecones among three Christmas wreaths?

Problem 2

Mommy gave Andy three Christmas stockings each having a different pattern: Snowman, Reindeer, and Penguin. Andy is to hang the stockings in the kitchen, the bedroom, and the living room, one stocking for each room. He is trying to decide which stocking to hang in which room. In how many different ways can he do this?

Problem 1 solution
A keen student may ask: are the wreaths different; are the pinecones different? These are great questions and very legitimate ones. Let's assume that all the pinecones are the same, and all the wreaths are the same. Let's make a table with three columns, each column for each wreath and write down the number of pinecones on each wreath. Although we label the wreaths for easy reference, we don't distinguish them, i.e. it doesn't matter whether all the pinecones end up on wreath 3 or wreath 1 or wreath 2.

Wreath 1	Wreath 2	Wreath 3
0	0	6
0	1	5
0	2	4
0	3	3
1	2	3
2	2	2

So in total, there are 6 different ways to divide 6 identical pinecones among 3 identical wreaths.
If the wreath are all different or the pinecones are different, then there will be more ways to do this.

Problem 2 solution
Let's denote Snowman as S, Reindeer as R, and Penguine as P. We can explicitly list down all different combinations in the following table.

Kitchen	**Living room**	**Bedroom**
S	R	P
S	P	R
R	S	P
R	P	S
P	S	R
P	R	S

See how we start with one of the patterns, and then exchange the position of the second and third items. Being systematic is very important so as not to miss a solution or double count.

Count how many times each pattern appears in each room. Does this make sense to you?

There are only a few objects in the previous problems: 6 pinecones, 3 wreaths, 3 rooms, 3 stockings. But imagine when we have a large number of objects, are we going to manually count all different arrangements? Let's look at the following problem:

A princess had 7 pairs of shoes in different colors to wear one on each day of the week. She liked to change the sequence of shoes from one week to the next. For example, if this she wears Pink, Blue, White, Red, Orange, Green, and Yellow from Monday to Sunday, then the following week she might wear them in this order: Blue, White, Red, Orange, Yellow, Green, and Pink. After how many weeks would she repeat the same sequence?

7 is not a large number so there probably won't be many different sequences. Every kid thought so when I gave them this problem but the answer always surprised them. It would take the princess more than 5,000 weeks, that is close to 100 years to repeat the same sequence of shoes that she wears in a week.

Obviously we are not going to manually write down all the sequences as there are too many of them.. Instead we analyze the problem with a smaller number of shoes: 2, 3, 4 pairs, then find a pattern and generalize it to a rule or a formula. That is what mathematicians do: they develop formulas that would solve a whole class of problems with arbitrarily large numbers.

The subject of counting the number of ways to arrange a certain number of objects or events according to various requirements is called combinatorics. It is not a subject commonly taught in school but one that has wide ranging applications. In college, it is needed for many courses such as probability, computer science, statistical mechanics. But when students are given combinatoric formulas without understanding where they come from, they don't really know how to use them. It is much better for the students to learn combinatorics first by solving problems and develop formulas themselves.

From my experience, the good news is that kids are not afraid of combinatorics problems and actually enjoy them. There are appropriate combinatorics problems for all ages starting from elementary school. Introduce them gradually as many simple sounding combinatorics problems turn out very difficult. These problems are also very popular in math competitions.

Finally, let's apply what we learn in problem 1 and problem 2 to solve the following problem:

Problem 3
Find all three-digit numbers each of which has the sum of its digits equal to 6, for example the number 132.

Problem 3 solution

The solution to problem 1 can give us a head start. Since the total number of pinecones is always 6, we can think of the number of pinecones in each wreath as one digit in a 3-digit number that we are looking for. The next step is to form as many 3-digits numbers as we can given 3 digits.

Digits	Numbers
0 , 0, 6	600
0, 1, 5	105 150 501 510
0, 2, 4	204 240 420 402
0, 3, 3	303 330
1, 2, 3	123 132 213 231 312 321
2, 2, 2	222

So in total there are sixteen 3-digit numbers that satisfy the requirement.
Notice that problem number 2 is disguised in the case of three digits 1, 2, 3. With three non-zero, distinguished digits, we can form six 3-digit numbers, similar to six different ways we can put three stockings in three rooms.